Existence of Positive Solutions for Boundary Value Problems of Second Order Difference Equations

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Abstract

We prove the existence of positive solutions for boundary value problems of second - order difference equations.

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1 Introduction

We investigate the existence of positive solutions for the boundary value problems of a second order difference equation of the form

$$\begin{cases} -\Delta[p(n-1)\Delta y(n-1)] + q(n)y(n) = f(n, y(n)), \quad n \in [a, b], \\ \alpha y(a-1) - \beta y^{[\Delta]}(a-1) = 0, \\ \gamma y(b) + \delta y^{[\Delta]}(b) = 0, \end{cases} \tag{1.1}$$

where a, b $(b > a+1)$ are integers and $[a, b]$ denote the discrete set $\{a, a+1, ..., b\}$. As usual, Δ denotes the forward difference operator and the quasi Δ - derivative of $y(n)$ defined by

$$y^{[\Delta]}(n) = p(n)\Delta y(n).$$

We will assume that the following conditions are satisfied.

(H1) $p(n) > 0, \;\; q(n) \geq 0.$

(H2) $\alpha, \beta, \gamma, \delta \geq 0, \quad \alpha + \beta > 0, \quad \gamma + \delta > 0;$

if $q(n) \equiv 0$ $(a \leq n \leq b)$, then $\alpha + \gamma > 0$.

(H3) $f : [a, b] \times \mathbf{R} \to \mathbf{R}$ is continuous with respect to y and $f(n, y) \geq 0$ for $y \in \mathbf{R}^+$, where $\mathbf{R}^+$ denotes the set of nonnegative real numbers.

In this paper, we shall use a fixed point index theorem in cones to investigate the existence of positive solutions to BVP (1.1).

2 Main Theorems

Denote by $\varphi(n)$ and $\psi(n)$ the solutions of the corresponding homogeneous equation

$$-\triangle[p(n-1)\triangle y(n-1)] + q(n)y(n) = 0, \quad n \in [a, b], \tag{2.1}$$

under the initial conditions

$$\varphi(a-1) = \beta, \quad \varphi^{[\triangle]}(a-1) = \alpha; \tag{2.2}$$

$$\psi(b) = \delta, \quad \psi^{[\triangle]}(b) = -\gamma. \tag{2.3}$$

Define the number D by

$$D := \alpha\psi(a-1) - \beta\psi^{[\triangle]}(a-1) = \gamma\varphi(b) + \delta\varphi^{[\triangle]}(b). \tag{2.4}$$

Using the initial conditions (2.2) and (2.3), we can deduce from equation (2.1) for $\varphi(n)$ and $\psi(n)$ the following equations:

$$\varphi(n) = \beta + \alpha \sum_{k=a-1}^{n-1} \frac{1}{p(k)} + \sum_{\tau=a}^{n-1} \left[\sum_{k=\tau}^{n-1} \frac{1}{p(k)} \right] q(\tau)\varphi(\tau). \tag{2.5}$$

$$\psi(n) = \delta + \gamma \sum_{k=n}^{b-1} \frac{1}{p(k)} + \sum_{\tau=n+1}^{b} \left[\sum_{k=n}^{\tau-1} \frac{1}{p(k)} \right] q(\tau)\psi(\tau). \tag{2.6}$$

(See [1].) Let $G(n, s)$ be the Green's function for the boundary value problem

$$\begin{cases} -\Delta[p(n-1)\Delta y(n-1)] + q(n)y(n) = 0, \quad n \in [a, b], \\ \alpha y(a-1) - \beta y^{[\Delta]}(a-1) = 0, \\ \gamma y(b) + \delta y^{[\Delta]}(b) = 0, \end{cases}$$

is given by

$$G(n,s) = \frac{1}{D}\begin{cases} \varphi(n)\psi(s), & a-1 \le n \le s \le b+1, \\ \\ \varphi(s)\psi(n), & a-1 \le s \le n \le b+1 \end{cases} \tag{2.7}$$

where $\varphi(n)$ and $\psi(n)$ are given in (2.5) and (2.6), respectively, and it is obvious from (H2) that $D > 0$ holds.

Suppose that $y(n)$ is a solution of the BVP (1.1), then it could be expressed as

$$y(n) = \sum_{s=a}^{b} G(n,s) f(s, y(s)), \quad a \le n \le b. \tag{2.8}$$

Furthermore, a solution $y(n)$ of (1.1) is called a positive solution if $y(n) > 0$ for $n \in [a,b]$.

Lemma 2.1 *(See [1].) Assume that condition (H2) is satisfied. Then :*

(i) $0 \le G(n,s) \le G(s,s)$ *for* $n, s \in [a-1, b]$ *;*

(ii) $G(n,s) \ge \sigma G(s,s)$ *for* $n \in [a, b-1]$ *and* $s \in [a-1, b]$*,*

where

$$\sigma = min\{I_1, I_2\} \tag{2.9}$$

in which

$$I_1 = \left\{\beta + \frac{\alpha}{p(a-1)}\right\}\left\{\beta + \alpha \sum_{k=a-1}^{b-1} \frac{1}{p(k)} + \sum_{\tau=a}^{b-1}\left[\sum_{k=\tau}^{b-1}\frac{1}{p(k)}\right] q(\tau)\varphi(\tau)\right\}^{-1},$$

$$I_2 = \left\{\delta + \frac{\gamma + \delta q(b)}{p(b-1)}\right\}\left\{\delta + \gamma \sum_{k=a-1}^{b-1} \frac{1}{p(k)} + \sum_{\tau=a}^{b}\left[\sum_{k=a-1}^{\tau-1}\frac{1}{p(k)}\right] q(\tau)\psi(\tau)\right\}^{-1}.$$

Note that the number σ defined by (2.9) satisfies the inequalities $0 < \sigma < 1$.

Lemma 2.2 *(See [2,3,4,5].) Assume that* $\mathcal{B}$ *is a Banach space, and* $\mathcal{K} \subset \mathcal{B}$ *is a cone in* $\mathcal{B}$*. Let* $\mathcal{K}_p = \{y \in \mathcal{K} : \|y\| < p\}$*. Furthermore, assume that* $\Phi : \mathcal{K} \to \mathcal{K}$ *is a compact map and* $\Phi y \ne y$ *for* $y \in \partial\mathcal{K}_p = \{y \in \mathcal{K} : \|y\| = p\}$*. Then one has the following conclusions:*

1. *If* $\|y\| \le \|\Phi y\|$ *for* $y \in \partial\mathcal{K}_p$*, then* $i(\Phi, \mathcal{K}_p, \mathcal{K}) = 0$*;*

2. *If* $\|y\| \ge \|\Phi y\|$ *for* $y \in \partial\mathcal{K}_p$*, then* $i(\Phi, \mathcal{K}_p, \mathcal{K}) = 1$*.*

Let $\mathcal{K}$ be a cone in the $b-a+1$ dimensional real Banach space $\mathcal{B}$ of real-valued functions $y(n)$ defined on $[a,b]$ by

$$\mathcal{K} = \{y \in \mathcal{B} : \min_{a \le n \le b-1} y(n) \ge \sigma \|y\|\},$$

where $\|y\| := \max_{a \le n \le b} |y(n)|$ and σ is defined by (2.9).

Define an operator $\Phi : \mathcal{K} \to \mathcal{K}$ by

$$(\Phi y)(n) = \sum_{s=a}^{b} G(n,s) f(s, y(s)), \quad a \le n \le b \tag{2.10}$$

Lemma 2.3 $\Phi(\mathcal{K}) \subset \mathcal{K}$.

Proof. It follows from the definition of $\mathcal{K}$ and Lemma 2.1 that

$$\begin{aligned}
\min_{a \le n \le b-1} (\phi y)(n) &= \min_{a \le n \le b-1} \sum_{s=a}^{b} G(n,s) f(s, y(s)) \\
&\ge \sigma \sum_{s=a}^{b} G(s,s) f(s, y(s)) \\
&\ge \sigma \max_{a \le n \le b} \sum_{s=a}^{b} G(n,s) f(s, y(s)) \\
&= \sigma \|\Phi y\|,
\end{aligned}$$

which implies $\Phi(\mathcal{K}) \subset \mathcal{K}$.

Lemma 2.4 $\Phi : \mathcal{K} \to \mathcal{K}$ *is completely continuous.*

Lemma 2.5 *If*

$$\lim_{v \to 0^+} \frac{f(n,v)}{v} = \infty \text{ and } \lim_{v \to +\infty} \frac{f(n,v)}{v} = \infty, \tag{2.11}$$

for all $n \in [a,b]$*, then there exist* $0 < r_0 < R_0 < \infty$ *such that* $i(\Phi, \mathcal{K}_r, \mathcal{K}) = 0$ *for* $0 < r \le r_0$ *and* $i(\Phi, \mathcal{K}_R, \mathcal{K}) = 0$ *for* $R \ge R_0$.

Proof. Choose $M > 0$ such that

$$\sigma^2 M \sum_{s=a}^{b-1} G(s,s) > 1. \tag{2.12}$$

By using the first equality of (2.11) we can choose $r_0 > 0$ such that

$$f(n,v) \geq Mv, \quad 0 \leq v \leq r_0,$$

If $u \in \partial\mathcal{K}_r$ $(0 < r \leq r_0)$, then for $n_0 \in [a, b-1]$, we obtain

$$(\Phi y)(n_0) = \sum_{s=a}^{b} G(n_0, s) f(s, y(s))$$

$$\geq \sigma \sum_{s=a}^{b-1} G(s,s) f(s, y(s))$$

$$\geq \sigma M \sum_{s=a}^{b-1} G(s,s) y(s)$$

$$\geq \sigma^2 \|y\| M \sum_{s=a}^{b-1} G(s,s)$$

$$> \|y\|.$$

This leads to

$$\|\Phi y\| > \|y\|, \quad \forall y \in \partial\mathcal{K}_r.$$

Thus we have from Lemma 2.2 $i(\Phi, \mathcal{K}_r, \mathcal{K}) = 0$, for $0 < r \leq r_0$. On the other hand, the second equality of (2.11) implies that for every $M > 0$, there is an $R_0 > r_0$ such that

$$f(n,v) \geq Mv, \quad v \geq \sigma R_0; \tag{2.13}$$

here we choose $M > 0$ satisfying (2.12). For $y \in \partial\mathcal{K}_R$, $R \geq R_0$, we have from the definition of $\mathcal{K}_R$ that

$$y(n) \geq \sigma\|y\| = \sigma R, \ \ n \in [a, b-1].$$

Thus we have from (2.13) that

$$(\Phi y)(n_0) = \sum_{s=a}^{b} G(n_0, s) f(s, y(s))$$

$$\geq \sigma \sum_{s=a}^{b-1} G(s,s) f(s, y(s))$$

$$\geq \sigma^2 M R \sum_{s=a}^{b-1} G(s,s)$$

$$> R = \|y\|,$$

which leads to

$$\|\Phi y\| > \|y\|, \quad \forall y \in \partial\mathcal{K}_R.$$

Thus $i(\Phi, \mathcal{K}_R, \mathcal{K}) = 0$ for $R \geq R_0$. The proof is completed.

In the next theorem we will also assume the following condition on $f(n, v)$.

$$\textbf{(H4)} \qquad \lim_{v\to 0^+} \inf \min_{n\in[a,b]} \frac{f(n,v)}{v} > k\lambda_1, \quad \lim_{v\to+\infty} \sup \max_{n\in[a,b]} \frac{f(n,v)}{v} < q\lambda_1;$$

where $k > 0$ is large enough such that

$$k\sigma \sum_{s=a}^{b-1} \phi_1(n) \geq \sum_{n=a}^{b} \phi_1(n),$$

and $q > 0$ is small enough such that

$$\sigma \sum_{n=a}^{b-1} \phi_1(n) \geq q \sum_{n=a}^{b} \phi_1(n);$$

where $\phi_1(n)$ ($\phi_1(n) > 0, n \in [a, b]$) is the eigenfunction related to the smallest eigenvalue λ_1 ($\lambda_1 > 0$) of the eigenvalue problem

$$-\Delta[p(n-1)\Delta\phi_1(n-1)] + q(n)\phi_1(n) = \lambda\phi_1(n),$$

$$\alpha\phi_1(a-1) - \beta\phi_1^{[\Delta]}(a-1) = 0, \quad \gamma\phi_1(b) + \delta\phi_1^{[\Delta]}(b) = 0.$$

Theorem 2.1 *Assume that conditions (H1)-(H4) are satisfied. Then the BVP (1.1) has at least one positive solution.*

Proof. Fix $0 < m < 1 < m_1$ and let $f_1(y) = y^m + y^{m_1}$ for $y \geq 0$. Then $f_1(y)$ satisfies (2.11). Define $\Phi_1 : \mathcal{K} \to \mathcal{K}$ by

$$(\Phi_1 y)(n) = \sum_{s=a}^{b} G(n,s) f_1(y(s)), \quad a \leq n \leq b \tag{2.14}$$

Then by using Lemma 2.5, we conclude that there exist $0 < r_0 < R_0 < \infty$, such that

$$0 < r \leq r_0 \text{ implies } i(\Phi_1, \mathcal{K}_r, \mathcal{K}) = 0, \tag{2.15}$$

and

$$R \geq R_0 \text{ implies } i(\Phi_1, \mathcal{K}_R, \mathcal{K}) = 0. \tag{2.16}$$

Define $H : [0,1] \times \mathcal{K} \to \mathcal{K}$ by $H(t,y) = (1-t)\Phi y + t\Phi_1 y$, then H is a completely continuous operator. By the first inequality in (H4) and the definition of f_1, there are $\varepsilon > 0$ and $0 < r_1 \leq r_0$ such that

$$f(n,y) \geq (k\lambda_1 + \varepsilon)y, \quad \forall n \in [a,b],\, 0 \leq y \leq r_1, \tag{2.17}$$

$$f_1(y) \geq (k\lambda_1 + \varepsilon)y, \quad \forall\, 0 \leq y \leq r_1.$$

We now prove that $H(t,y) \neq y$ for all $t \in [0,1]$ and $y \in \partial\mathcal{K}_{r_1}$. In fact if there exists $t_0 \in [0,1]$ and $y_1 \in \partial\mathcal{K}_{r_1}$ such that $H(t_0, y_1) = y_1$, then $y_1(n)$ satisfies the equation

$$-\Delta[p(n-1)\Delta y_1(n-1)] + q(n)y_1(n) = (1-t_0)f(n, y_1(n) + t_0 f_1(y_1(n)), \quad a \leq n \leq b$$

and the boundary condition. Multiplying the last equation by $\phi_1(n)$ and then summing it from a to b, using summation by parts in the left hand side two times, we get that

$$\lambda_1 \sum_{n=a}^{b} \phi_1(n)y_1(n) = \sum_{n=a}^{b}[(1-t_0)f(n,y_1(n)) + t_0 f_1(y_1(n))]\phi_1(n) \tag{2.18}$$

$$\geq \sum_{n=a}^{b-1}[(1-t_0)f(n,y_1(n)) + t_0 f_1(y_1(n))]\phi_1(n),$$

we obtain from (2.17) that

$$\geq \sum_{n=a}^{b-1}[(1-t_0)(k\lambda_1+\varepsilon)y_1(n) + t_0(k\lambda_1+\varepsilon)y_1(n)]\phi_1(n)$$

$$= (\lambda_1 + \frac{\varepsilon}{k})k \sum_{n=a}^{b-1} \phi_1(n)y_1(n)$$

$$\geq (\lambda_1 + \frac{\varepsilon}{k})k\sigma\|y_1\| \sum_{n=a}^{b-1} \phi_1(n)$$

$$\geq (\lambda_1 + \frac{\varepsilon}{k})\|y_1\| \sum_{n=a}^{b} \phi_1(n). \tag{2.19}$$

We also have

$$\lambda_1 \sum_{n=a}^{b} \phi_1(n)y_1(n) \leq \lambda_1\|y_1\| \sum_{n=a}^{b} \phi_1(n). \tag{2.20}$$

Which together with (2.19) leads to

$$\lambda_1 \geq \lambda_1 + \frac{\varepsilon}{k}.$$

This is impossible. Thus $H(t,y) \neq y$ for $y \in \partial\mathcal{K}_{r_1}$ and $t \in [0,1]$. By (2.15) and the homotopy invariance of the fixed point index (see [6]), we get that

$$\begin{aligned} i(\Phi, \mathcal{K}_{r_1}, \mathcal{K}) &= i(H(0,\cdot), \mathcal{K}_{r_1}, \mathcal{K}) \\ &= i(H(1,\cdot), \mathcal{K}_{r_1}, \mathcal{K}) \\ &= i(\Phi_1, \mathcal{K}_{r_1}, \mathcal{K}) \\ &= 0. \end{aligned} \tag{2.21}$$

On the other hand, according to the second inequality of (H4), there exist $\epsilon > 0$ and $R' > R_o$ such that

$$f(n,y) \leq (q\lambda_1 - \varepsilon)y, \quad y > R' \text{ and } n \in [a,b].$$

Set

$$\mathcal{C} = \max_{a \leq n \leq b, 0 \leq y \leq R'} |f(n,y) - (q\lambda_1 - \varepsilon)y| + 1, \text{ then we deduce that}$$

$$f(n,y) \leq (q\lambda_1 - \varepsilon)y + \mathcal{C}, \quad y \geq 0 \text{ and } n \in [a,b]. \tag{2.22}$$

Define $H_1 : [0,1] \times \mathcal{K} \to \mathcal{K}$ by $H_1(t,y) = t\Phi y$, then H_1 is a completely continuous operator.

We now prove that there exists $R_1 \geq R'$ such that $H_1(t,y) \neq y$ for any $0 \leq t \leq 1$ and $y \in \mathcal{K}$, $\|y\| \geq R_1$. In fact, if $0 \leq t_0 \leq 1$ and $y_1 \in \mathcal{K}$ satisfy $H_1(t_0, y_1) = y_1$, then

$$\begin{aligned} \lambda_1 \sum_{n=a}^{b} y_1(n)\phi_1(n) &\leq \sum_{n=a}^{b} f(n, y_1(n))\phi_1(n) \\ &\leq q(\lambda_1 - \frac{\varepsilon}{q})\|y_1\| \sum_{n=a}^{b} \phi_1(n) + \mathcal{C} \sum_{n=a}^{b} \phi_1(n), \end{aligned} \tag{2.23}$$

and

$$\lambda_1 \sum_{n=a}^{b} y_1(n)\phi_1(n) \geq \lambda_1 \sum_{n=a}^{b-1} y_1(n)\phi_1(n)$$

$$\geq \lambda_1 \sigma \|y_1\| \sum_{n=a}^{b-1} \phi_1(n)$$

$$\geq \lambda_1 q \|y_1\| \sum_{n=a}^{b-1} \phi_1(n), \tag{2.24}$$

Combining (2.23) with (2.24), we have

$$\|y_1\| \leq \frac{\mathcal{C}}{\varepsilon} = \tilde{R}_1.$$

Hence if

$$R_1 = \max\{R', \tilde{R}_1\} + 1$$

then we have that

$$H_1(t, y) \neq y \text{ for } t \in [0, 1]\ ,\ y \in \mathcal{K},\ \|y\| \geq R_1.$$

Therefore we have by the homotopy invariance of the fixed point index

$$\begin{aligned} i(\Phi, \mathcal{K}_{R_1}, \mathcal{K}) &= i(H_1(1, \cdot), \mathcal{K}_{R_1}, \mathcal{K}) \\ &= i(H_1(0, \cdot), \mathcal{K}_{R_1}, \mathcal{K}) \\ &= i(\Theta, \mathcal{K}_{R_1}, \mathcal{K}) \\ &= 1, \end{aligned} \tag{2.25}$$

where Θ is zero operator . Use (2.21) and (2.25) to conclude that

$$i(\Phi, \mathcal{K}_{R_1} \backslash \mathcal{K}_{r_1}, \mathcal{K}) = i(\Phi, \mathcal{K}_{R_1}, \mathcal{K}) - i(\Phi, \mathcal{K}_{r_1}, \mathcal{K}) = 1 - 0 = 1.$$

Therefore Φ has a fixed point in $(\mathcal{K}_{R_1} \backslash \mathcal{K}_{r_1})$.

(H5) $\lim_{v \to 0^+} \sup \max_{n \in [a,b]} \frac{f(n, v)}{v} < q\lambda_1,$

$$\lim_{v \to +\infty} \inf \min_{n \in [a,b]} \frac{f(n, v)}{v} > k\lambda_1.$$

Theorem 2.2 *Assume that conditions (H1)-(H3) and (H5) are satisfied. Then the BVP (1.1) has at least one positive solution.*

Proof. Define $H_1 : [0,1] \times \mathcal{K} \to \mathcal{K}$ by $H_1(t,y) = t\Phi y$, then H_1 is a completely continuous operator. By the first inequality in (H5), there exist $\varepsilon > 0$ and $r_1 : 0 < r_1 \leq r_0$ such that

$$f(n,v) \leq (q\lambda_1 - \varepsilon)v, \quad \forall n \in [a,b],\, 0 \leq v \leq r_1. \tag{2.26}$$

We now prove that $H_1(t,y) \neq y$ for $0 \leq t \leq 1$ and $y \in \partial\mathcal{K}_{r_1}$. In fact, if there exists $0 \leq t_0 \leq 1$ and $y_1 \in \partial\mathcal{K}_{r_1}$, such that $H_1(t_0,y_1) = y_1$, then the $y_1(n)$ satisfy the boundary condition.

$$-\Delta[p(n-1)\Delta y_1(n-1)] + q(n)y_1(n) = t_0 f(n, y_1(n)), \quad \forall n \in [a,b].$$

Multiplying the last equality by $\phi_1(n)$ and summing from a to b, , we see that

$$\lambda_1 \sum_{n=a}^{b} y_1(n)\phi_1(n) = t_0 \sum_{n=a}^{b} f(n, y_1(n))\phi_1(n) \tag{2.27}$$

$$\leq \sum_{n=a}^{b} f(n, y_1(n))\phi_1(n)$$

$$\leq (q\lambda_1 - \varepsilon)\|y_1\| \sum_{n=a}^{b} \phi_1(n), \tag{2.28}$$

and

$$\lambda_1 \sum_{n=a}^{b} y_1(n)\phi_1(n) \geq \lambda_1 \sum_{n=a}^{b-1} y_1(n)\phi_1(n)$$

$$\geq \lambda_1 \sigma \|y_1\| \sum_{n=a}^{b-1} \phi_1(n)$$

$$\geq \lambda_1 q \|y_1\| \sum_{n=a}^{b} \phi_1(n), \tag{2.29}$$

which together with (2.28) lead to

$$\lambda_1 q \leq \lambda q_1 - \varepsilon.$$

This is impossible. Using homotopy invariance of the fixed point index, we have that

$$i(\Phi, \mathcal{K}_{r_1}, \mathcal{K}) = i(H_1(0,\cdot), \mathcal{K}_{r_1}, \mathcal{K})$$

$$= i(\Theta, \mathcal{K}_{r_1}, \mathcal{K})$$

$$= 1. \tag{2.30}$$

Define $H : [0,1] \times \mathcal{K} \to \mathcal{K}$ by $H(t,y) = (1-t)\Phi y + t\Phi_1 y$, then H is a completely continuous operator. By the second inequality in (H5) and the definition of f_1, there exist $\varepsilon > 0$ and $R' > R_0$ such that

$$f(n,y) \geq (k\lambda_1 + \varepsilon)y, \quad \forall n \in [a,b],\ y \geq R',$$

$$f_1(y) \geq (k\lambda_1 + \varepsilon)y, \quad \forall y \geq R'.$$

Let

$$\mathcal{C} = \max_{0 \leq y \leq R', n \in [a,b]} |f(n,y) - (k\lambda_1 + \varepsilon)y| + \max_{0 \leq y \leq R'} |f_1(y) - (k\lambda_1 + \varepsilon)y| + 1,$$

then, it is obvious that

$$f(n,y) \geq (k\lambda_1 + \varepsilon)y - c, \quad \forall n \in [a,b], y \geq 0, \tag{2.31}$$

$$f_1(y) \geq (k\lambda_1 + \varepsilon)y - c, \quad \forall\ y \geq 0. \tag{2.32}$$

We now prove that there exists $R_1 \geq R'$ such that $H(t,y) \neq y$ for any $0 \leq t \leq 1$ and $y \in \mathcal{K}, \|y\| \geq R_1$. In fact, if $0 \leq t_0 \leq 1$ and $y_1 \in \mathcal{K}$ satisfying $H(t_0, y_1) = y_1$, then using (2.31) and (2.32), it is analogous to the argument of (2.19) and (2.20) that

$$\begin{aligned}
\lambda_1 \sum_{n=a}^{b} \phi_1(n) y_1(n) &= \sum_{n=a}^{b} [(1-t_0) f(n, y_1(n)) + t_0 f_1(y_1(n))] \phi_1(n) \\
&\geq \sum_{n=a}^{b-1} [(1-t_0) f(n, y_1(n)) + t_0 f_1(y_1(n))] \phi_1(n) \\
&\geq \sum_{n=a}^{b-1} \{(1-t_0)[(k\lambda_1 + \varepsilon) y_1(n) - c] + t_0[(k\lambda_1 + \varepsilon) y_1(n) \\
&\quad - c]\} \phi_1(n) \\
&= \sum_{n=a}^{b-1} [(k\lambda_1 + \varepsilon) y_1(n) - c] \phi_1(n) \\
&\geq (\lambda_1 + \frac{\varepsilon}{k}) k\sigma \|y_1\| \sum_{n=a}^{b-1} \phi_1(n) - c \sum_{n=a}^{b-1} \phi_1(n)
\end{aligned} \tag{2.33}$$

$$\lambda_1 \sum_{n=a}^{b} \phi_1(n) y_1(n) \leq \lambda_1 \|y_1\| \sum_{n=a}^{b} \phi_1(n)$$

$$\leq \lambda_1 \|y_1\| k\sigma \sum_{n=a}^{b-1} \phi_1(n). \tag{2.34}$$

(2.33) and (2.34) lead to $\|y_1\| \leq \dfrac{c}{\varepsilon\sigma} = \tilde{R_1}$. Let $R_1 = \max\{R', \tilde{R_1}\} + 1$.

We obtain

$$\begin{aligned} i(\Phi, \mathcal{K}_{R_1}, \mathcal{K}) &= i(H(0,\cdot), \mathcal{K}_{R_1}, \mathcal{K}) \\ &= i(H(1,\cdot), \mathcal{K}_{R_1}, \mathcal{K}) \\ &= i(\Phi_1, \mathcal{K}_{R_1}, \mathcal{K}) \\ &= 0. \end{aligned} \tag{2.35}$$

Use (2.30) and (2.35) to conclude that

$$i(\Phi, \mathcal{K}_{R_1} \backslash \mathcal{K}_{r_1}, \mathcal{K}) = -1.$$

Therefore Φ has a fixed point in $\mathcal{K}_{R_1} \backslash \mathcal{K}_{r_1}$.

Corollary. Using the following (H6) or (H7) instead of (H4) or (H5) the conclusion of Theorem 2.1 and Theorem 2.2 are true. For all $n \in [a, b]$,

(H6) $\displaystyle\lim_{v\to 0^+} \frac{f(n,v)}{v} = +\infty, \quad \lim_{v\to +\infty} \frac{f(n,v)}{v} = 0$ (sublinear);

(H7) $\displaystyle\lim_{v\to 0^+} \frac{f(n,v)}{v} = 0, \quad \lim_{v\to +\infty} \frac{f(n,v)}{v} = +\infty$ (superlinear).

References

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